Optimization for Machine Learning
CS-439

Lecture 10: Accelerated Gradient Descent, Gradient-free, and Applications

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Chapter X.1

Accelerated Gradient Descent

Smooth convex functions: less than $\mathcal{O}(1/\varepsilon)$ steps?

Fixing L and $R = \|\mathbf{x}_0 - \mathbf{x}^\star\|$, the error of gradient descent after T steps is $O(1/T)$.

Lee and Wright [LW19]:

- A better upper bound of $o(1/T)$ holds.
- A lower bound of $\Omega(1/T^{1+\delta})$ also holds, for any fixed $\delta > 0$.

So, gradient descent is slightly faster on smooth functions than what we proved, but not significantly.

First-order methods: less than $\mathcal{O}(1/\varepsilon)$ steps?

Maybe gradient descent is not the best possible algorithm?

After all, it is just some algorithm that uses gradient information.

First-order method:

- An algorithm that gains access to f only via an oracle that is able to return values of f and ∇f at arbitrary points.
- Gradient descent is a specific first-order method.

What is the best first-order method for smooth convex functions, the one with the smallest upper bound on the number of oracle calls in the worst case?

Nemirovski and Yudin 1979 [NY83]: every first-order method needs in the worst case $\Omega(1/\sqrt{\varepsilon})$ steps (gradient evaluations) in order to achieve an additive error of ε on smooth functions.

There is a gap between $O(1/\varepsilon)$ (gradient descent) and the lower bound!

Acceleration for smooth convex functions: $\mathcal{O}(1/\sqrt{\varepsilon})$ steps

Nesterov 1983 [Nes83, Nes18]: There is a first-order method that needs only $O(1/\sqrt{\varepsilon})$ steps on smooth convex functions, and by the lower bound of Nemirovski and Yudin, this is a best possible algorithm!

The algorithm is known as (Nesterov's) accelerated gradient descent.

A number of (similar) optimal algorithms with other proofs of the $\mathcal{O}(1/\sqrt{\varepsilon})$ upper bound are known, but there is no well-established "simplest proof".

Here: a recent proof based on potential functions [BG17]. Proof is simple but not very instructive (it works, but it's not clear why).

Nesterov's accelerated gradient descent

Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex, differentiable, and smooth with parameter L. Choose $\mathbf{z}_0 = \mathbf{y}_0 = \mathbf{x}_0$ arbitrary. For $t \geq 0$, set

$$\begin{aligned} \mathbf{y}_{t+1} &:= \mathbf{x}_t - \frac{1}{L}\nabla f(\mathbf{x}_t) \\ \mathbf{z}_{t+1} &:= \mathbf{z}_t - \frac{t+1}{2L}\nabla f(\mathbf{x}_t) \\ \mathbf{x}_{t+1} &:= \frac{t+1}{t+3}\mathbf{y}_{t+1} + \frac{2}{t+3}\mathbf{z}_{t+1}. \end{aligned}$$

- Perform a "smooth step" from $\mathbf{x}_t$ to $\mathbf{y}_{t+1}$.
- Perform a more aggressive step from $\mathbf{z}_t$ to $\mathbf{z}_{t+1}$.
- Next iterate $\mathbf{x}_{t+1}$ is a weighted average of $\mathbf{y}_{t+1}$ and $\mathbf{z}_{t+1}$, where we compensate for the more aggressive step by giving $\mathbf{z}_{t+1}$ a relatively low weight.

Why should this work??

Nesterov's accelerated gradient descent: Error bound

Theorem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex and differentiable with a global minimum $\mathbf{x}^\star$; furthermore, suppose that f is smooth with parameter L. Accelerated gradient descent yields

$$f(\mathbf{y}_T) - f(\mathbf{x}^\star) \leq \frac{2L \left\|\mathbf{z}_0 - \mathbf{x}^\star\right\|^2}{T(T+1)}, \quad T > 0.$$

To reach error at most ε, accelerated gradient descent therefore only needs $O(1/\sqrt{\varepsilon})$ steps instead of $O(1/\varepsilon)$.

Recall the bound for gradient descent:

$$f(\mathbf{x}_T) - f(\mathbf{x}^\star) \leq \frac{L}{2T}\|\mathbf{x}_0 - \mathbf{x}^\star\|^2, \quad T > 0.$$

Nesterov's accelerated gradient descent: The potential function

Idea: assign a potential $\Phi(t)$ to each time t and show that $\Phi(t+1) \leq \Phi(t)$.

Out of the blue: let's define the potential as

$$\Phi(t) := t(t+1)\left(f(\mathbf{y}_t) - f(\mathbf{x}^\star)\right) + 2L\left\|\mathbf{z}_t - \mathbf{x}^\star\right\|^2.$$

If we can show that the potential always decreases, we get

$$\underbrace{T(T+1)\left(f(\mathbf{y}_T) - f(\mathbf{x}^\star)\right) + 2L\left\|\mathbf{z}_T - \mathbf{x}^\star\right\|^2}_{\Phi(T)} \leq \underbrace{2L\left\|\mathbf{z}_0 - \mathbf{x}^\star\right\|^2}_{\Phi(0)}.$$

Rewriting this, we get the claimed error bound.

Potential function decrease: Three Ingredients

Sufficient decrease for the smooth step from $\mathbf{x}_t$ to $\mathbf{y}_{t+1}$:

$$f(\mathbf{y}_{t+1}) \leq f(\mathbf{x}_t) - \frac{1}{2L}\|\nabla f(\mathbf{x}_t)\|^2; \tag{1}$$

Vanilla analysis for the more aggressive step from $\mathbf{z}_t$ to $\mathbf{z}_{t+1}$: $(\gamma = \frac{t+1}{2L}$, $\mathbf{g}_t = \nabla f(\mathbf{x}_t))$:

$$\mathbf{g}_t^\top(\mathbf{z}_t - \mathbf{x}^\star) = \frac{t+1}{4L}\|\mathbf{g}_t\|^2 + \frac{L}{t+1}\left(\|\mathbf{z}_t - \mathbf{x}^\star\|^2 - \|\mathbf{z}_{t+1} - \mathbf{x}^\star\|^2\right); \tag{2}$$

Convexity (graph of f is above the tangent hyperplane at $\mathbf{x}_t$):

$$f(\mathbf{x}_t) - f(\mathbf{w}) \leq \mathbf{g}_t^\top(\mathbf{x}_t - \mathbf{w}), \quad \mathbf{w} \in \mathbb{R}^d. \tag{3}$$

Potential function decrease: Proof

By definition of potential,

$$\begin{aligned}\Phi(t+1) &= t(t+1)\left(f(\mathbf{y}_{t+1}) - f(\mathbf{x}^\star)\right) + 2(t+1)\left(f(\mathbf{y}_{t+1}) - f(\mathbf{x}^\star)\right) + 2L\left\|\mathbf{z}_{t+1} - \mathbf{x}^\star\right\|^2, \\ \Phi(t) &= t(t+1)\left(f(\mathbf{y}_t) - f(\mathbf{x}^\star)\right) + 2L\left\|\mathbf{z}_t - \mathbf{x}^\star\right\|^2.\end{aligned}$$

Now, prove that $\Delta := (\Phi(t+1) - \Phi(t))/(t+1) \leq 0$:

$$\begin{aligned}\Delta &= t\left(f(\mathbf{y}_{t+1}) - f(\mathbf{y}_t)\right) + 2\left(f(\mathbf{y}_{t+1}) - f(\mathbf{x}^\star)\right) + \frac{2L}{t+1}\left(\left\|\mathbf{z}_{t+1} - \mathbf{x}^\star\right\|^2 - \left\|\mathbf{z}_t - \mathbf{x}^\star\right\|^2\right) \\ &\overset{(2)}{=} t\left(f(\mathbf{y}_{t+1}) - f(\mathbf{y}_t)\right) + 2\left(f(\mathbf{y}_{t+1}) - f(\mathbf{x}^\star)\right) + \frac{t+1}{2L}\left\|\mathbf{g}_t\right\|^2 - 2\mathbf{g}_t^\top(\mathbf{z}_t - \mathbf{x}^\star) \\ &\overset{(1)}{\leq} t\left(f(\mathbf{x}_t) - f(\mathbf{y}_t)\right) + 2\left(f(\mathbf{x}_t) - f(\mathbf{x}^\star)\right) - \frac{1}{2L}\left\|\mathbf{g}_t\right\|^2 - 2\mathbf{g}_t^\top(\mathbf{z}_t - \mathbf{x}^\star) \\ &\leq t\left(f(\mathbf{x}_t) - f(\mathbf{y}_t)\right) + 2\left(f(\mathbf{x}_t) - f(\mathbf{x}^\star)\right) - 2\mathbf{g}_t^\top(\mathbf{z}_t - \mathbf{x}^\star) \\ &\overset{(3)}{\leq} t\mathbf{g}_t^\top(\mathbf{x}_t - \mathbf{y}_t) + 2\mathbf{g}_t^\top(\mathbf{x}_t - \mathbf{x}^\star) - 2\mathbf{g}_t^\top(\mathbf{z}_t - \mathbf{x}^\star) \\ &= \mathbf{g}_t^\top\left((t+2)\mathbf{x}_t - t\mathbf{y}_t - 2\mathbf{z}_t\right) \overset{\text{(algo)}}{=} \mathbf{g}_t^\top\mathbf{0} = 0. \quad \square\end{aligned}$$

Chapter X.2

Zero-Order Optimization

Look mom no gradients!

Can we optimize $\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x})$ if without access to gradients?

meet the newest fanciest optimization algorithm,...
Random search

pick a random direction $\mathbf{d}_t \in \mathbb{R}^d$

$$\gamma := \operatorname*{argmin}_{\gamma\in\mathbb{R}} f(\mathbf{x}_t + \gamma \mathbf{d}_t) \qquad \text{(line-search)}$$

$$\mathbf{x}_{t+1} := \mathbf{x}_t + \gamma \mathbf{d}_t$$

Convergence rate for derivative-free random search

Converges same as gradient descent - up to a slow-down factor d.

Proof. Assume that f is a L-smooth convex, differentiable function. For any γ, by smoothness, we have:

$$f(\mathbf{x}_t + \gamma \mathbf{d}_t) \leq f(\mathbf{x}_t) + \gamma \langle \mathbf{d}_t, \nabla f(\mathbf{x}_t) \rangle + \frac{\gamma^2 L}{2} \|\mathbf{d}_t\|^2$$

Minimizing the upper bound, there is a step size $\bar{\gamma}$ for which

$$f(\mathbf{x}_t + \bar{\gamma} \mathbf{d}_t) \leq f(\mathbf{x}_t) - \frac{1}{L} \left\langle \frac{\mathbf{d}_t}{\|\mathbf{d}_t\|}, \nabla f(\mathbf{x}_t) \right\rangle^2$$

The step size we actually took (based on f directly) can only be better:

$$f(\mathbf{x}_t + \gamma \mathbf{d}_t) \leq f(\mathbf{x}_t + \bar{\gamma} \mathbf{d}_t) \ .$$

Taking expectations, and using the Lemma $\mathbb{E}_{\mathbf{r}}(\mathbf{r}^\top \mathbf{g})^2 = \frac{1}{d} \|\mathbf{g}\|^2$ for $\mathbf{r} \sim$ sphere $\subseteq \mathbb{R}^d$:

$$\mathbb{E}[f(\mathbf{x}_t + \gamma \mathbf{d}_t)] \leq \mathbb{E}[f(\mathbf{x}_t)] - \frac{1}{Ld} \mathbb{E}[\|\nabla f(\mathbf{x}_t)\|^2] \ .$$

Convergence rate for derivative-free random search

Same as what we obtained for gradient descent,
now with an **extra factor of d**. d can be huge!!!

Can do the same for different function classes, as before

- For convex functions, we get a rate of $\mathcal{O}(dL/\varepsilon)$.
- For strongly convex, we get $\mathcal{O}(dL/\mu \log(1/\varepsilon))$.

Always d times the complexity of gradient descent on the function class.

credits to Moritz Hardt

Applications for derivative-free random search

Applications

- competitive method for **Reinforcement learning**
- memory and communication advantages: never need to store a gradient
- hyperparameter optimization, and other difficult e.g. discrete optimization problems

Reinforcement learning

$$\mathbf{s}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t, \mathbf{e}_t)\,.$$

where $\mathbf{s}_t$ is the state of the system, $\mathbf{a}_t$ is the control action, and $\mathbf{e}_t$ is some random noise. We assume that f is fixed, but unknown.

We search for a control 'policy'

$$\mathbf{a}_t := \pi(\mathbf{a}_1, \ldots, \mathbf{a}_{t-1}, \mathbf{s}_0, \ldots, \mathbf{s}_t)\,.$$

which takes a trajectory of the dynamical system and outputs a new control action. Want to maximize overall reward

$$\begin{aligned} \max_{\mathbf{a}_t} \ & \mathbb{E}_{\mathbf{e}_t}\Big[\sum_{t=0}^{N} R_t(\mathbf{s}_t, \mathbf{a}_t)\Big] \\ \text{s.t.} \ & \mathbf{s}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t, \mathbf{e}_t) \\ & (\mathbf{s}_0 \text{ given}) \end{aligned}$$

Examples: Simulations, Games (e.g. Atari), Alpha Go

Chapter X.3

Adaptive & other SGD Methods

Adagrad

Adagrad is an adaptive variant of SGD

pick a stochastic gradient $\mathbf{g}_t$

$$\text{update } [G_t]_i := \sum_{s=0}^{t} ([\mathbf{g}_s]_i)^2 \qquad \forall i$$

$$[\mathbf{x}_{t+1}]_i := [\mathbf{x}_t]_i - \frac{\gamma}{\sqrt{[G_t]_i}} [\mathbf{g}_t]_i \qquad \forall i$$

(standard choice of $\mathbf{g}_t := \nabla f_j(\mathbf{x}_t)$ for sum-structured objective functions $f = \sum_j f_j$)

- chooses an adaptive, coordinate-wise learning rate
- strong performance in practice
- Variants: Adadelta, Adam, RMSprop

Adam

Adam is a momentum variant of Adagrad

pick a stochastic gradient $\mathbf{g}_t$

$$\mathbf{m}_t := \beta_1 \mathbf{m}_{t-1} + (1-\beta_1)\mathbf{g}_t \qquad \text{(momentum term)}$$

$$[\mathbf{v}_t]_i := \beta_2 [\mathbf{v}_{t-1}]_i + (1-\beta_2)([\mathbf{g}_s]_i)^2 \quad \forall i \qquad \text{(2nd-order statistics)}$$

$$[\mathbf{x}_{t+1}]_i := [\mathbf{x}_t]_i - \frac{\gamma}{\sqrt{[\mathbf{v}_t]_i}}[\mathbf{m}_t]_i \quad \forall i$$

- faster forgetting of older weights
- momentum from previous gradients (see acceleration)
- (simplified version, without correction for initialization of $\mathbf{m}_0$,$\mathbf{v}_0$)
- strong performance in practice, e.g. for self-attention networks

SignSGD

Only use the sign (one bit) of each gradient entry:
SignSGD is a communication efficient variant of SGD.

pick a stochastic gradient $\mathbf{g}_t$

$$[\mathbf{x}_{t+1}]_i := [\mathbf{x}_t]_i - \gamma_t \, sign([\mathbf{g}_t]_i) \qquad \forall i$$

(with possible rescaling of γ_t with $\|\mathbf{g}_t\|_1$)

- communication efficient for distributed training
- convergence issues

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