Enhancing Dynamic Mode Decomposition using Autoencoder Networks

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Abstract #637

Abstract

Prediction, estimation, and control of dynamical systems remain challenging due to nonlinearity. The Koopman operator is an infinite-dimensional linear operator that evolves the observables of a dynamical system which we approximate by the dynamic mode decomposition (DMD) algorithm. Using DMD to predict the evolution of a nonlinear dynamical system over extended time horizons requires choosing the right observable function defined on the state space. A number of DMD modifications have been developed to choose the right observable function, such as Extended DMD. Here, we propose a simple machine learning based approach to find these coordinate transformations. This is done via a deep autoencoder network. The simple DMD autoencoder is tested and verified on nonlinear dynamical system time series datasets, including the classic pendulum and an approximation to fluid flow past a cylinder.

Keywords- Dynamic mode decomposition, Deep learning, Dynamical systems, Koopman analysis, Observable functions.

Introduction

Predictions of nonlinear dynamical systems is a fundamental problem in engineering. Whenever possible, it is desirable to work in a linear framework. Linear dynamical systems have closed form solutions. Moreover, there are many techniques for analyzing linear dynamical systems. The Dynamic Mode Decomposition is a computational method for mapping nonlinear time series into a linear representation via eigenfunctions of the Koopman Operator. Recent advances have shown that autoencoder networks can identify optimal coordinate transformations to approximate the Koopman Operator eigenfunctions [1]. Using this approach, we develop a robust equation free model to find global coordinate transformations which will enhance DMD long-term predictions.

Dynamic Mode Decomposition

Let x_t be the state vector of a nonlinear dynamical system. In order to create a linear model, our goal is to fit the dynamical system states to a model of the form:

$$\frac{d}{dt}x = Ax \quad \text{and} \quad x_{t+1} = Ax_t \tag{1}$$

Dynamic Mode Decomposition was developed by Schmid [4]. DMD is a dimensionality reduction algorithm [2]. Given a time series, the DMD computes the best fit operator A that advances the system measurements in time [2]. This is achieved by arranging the time series into two matrices, X and X':

$$X = \begin{bmatrix} | & | & & | \\ x_0 & x_1 & \dots & x_{m-1} \\ | & | & & | \end{bmatrix} \quad \text{and} \quad X' = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_m \\ | & | & & | \end{bmatrix} \tag{2}$$

Therefore, the best fit operator A is defined as $A = argmin_A ||X' - AX||_F$. By the singular value decomposition, $X \approx \tilde{U}\tilde{\Sigma}\tilde{V}^*$ where $\tilde{U} \in \mathbb{C}^{n\times r}$, $\tilde{\Sigma} \in \mathbb{C}^{r\times r}$, and $\tilde{V} \in \mathbb{C}^{m\times r}$. Therefore, the matrix A is obtained by $A = X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^*$.

There are many ways to measure the accuracy of the DMD fit. A simple way is to evaluate the following expression:

$$\begin{aligned} \|X' - AX\|_F &= \left\|X' - (X'V\Sigma^{-1}U^*)X\right\|_F \\ &= \left\|X' - (X'V\Sigma^{-1}U^*)(U\Sigma V^T)\right\|_F \\ &= \left\|X'(I - VV^T)\right\|_F \end{aligned} \tag{3}$$

Model Architecture

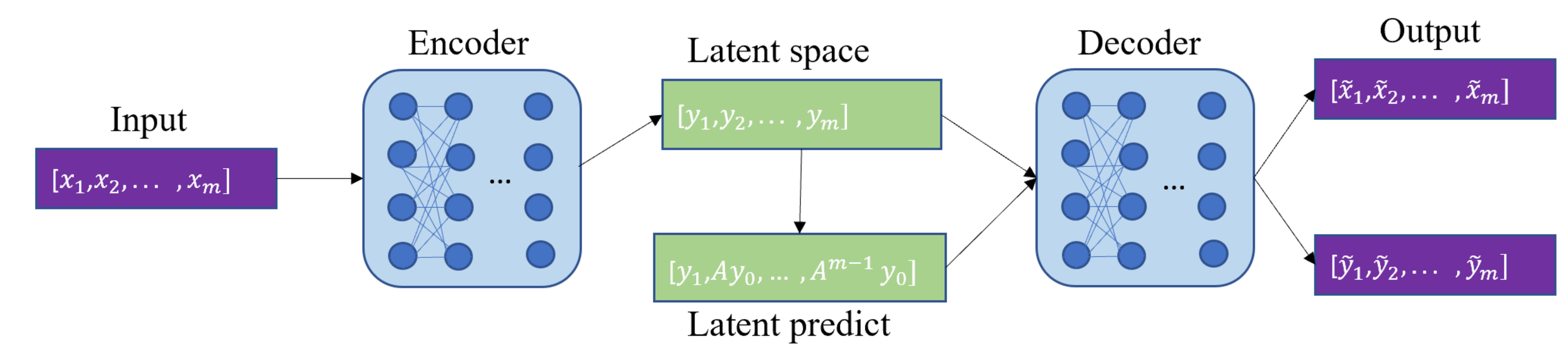

Figure 1: Simple DMD autoencoder architecture.

Figure 1 is an illustration of the simple DMD autoencoder architecture. The input time sequential dataset X is passed to the encoder which is a nonlinear mapping g. The latent space Y is predicted by the DMD, $\tilde{y}_{k+1} = A^k y_0$. Both the latent space Y and predicted latent space $\tilde{Y}$ are passed to the decoder g^{-1}. Lastly, the decoder outputs $g^{-1}(Y)$ and $g^{-1}(\tilde{Y})$.

Loss Function

The simple DMD autoencoder loss function is a combination of four evaluations:

1. Autoencoder reconstruction loss - ensure that the original dataset coordinates can be recovered.

$$L_1 = MSE\left\|X - \tilde{X}\right\| \tag{4}$$

2. DMD loss - evaluate the linearity of the latent space dynamics, based on equation (3).

$$L_2 = \left\|Y'(I - VV^T)\right\|_F \tag{5}$$

3. DMD reconstruction loss- evaluate the DMD least squares fit.

$$L_3 = MSE\left\|Y - \tilde{Y}\right\| \tag{6}$$

4. Linearity loss- evaluate the linearity of the latent space to enhance long-term predictions.

$$L_4 = MSE\left\|X - g^{-1}(\tilde{Y})\right\| \tag{7}$$

The final loss function is a weighted sum of the functions above: $L = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4 L_4$.

Results

Fluid Flow Past a Cylinder- the nonlinear mean-field model of fluid flow past a circular cylinder at Reynolds number 100, described by empirical Galerkin model [3].

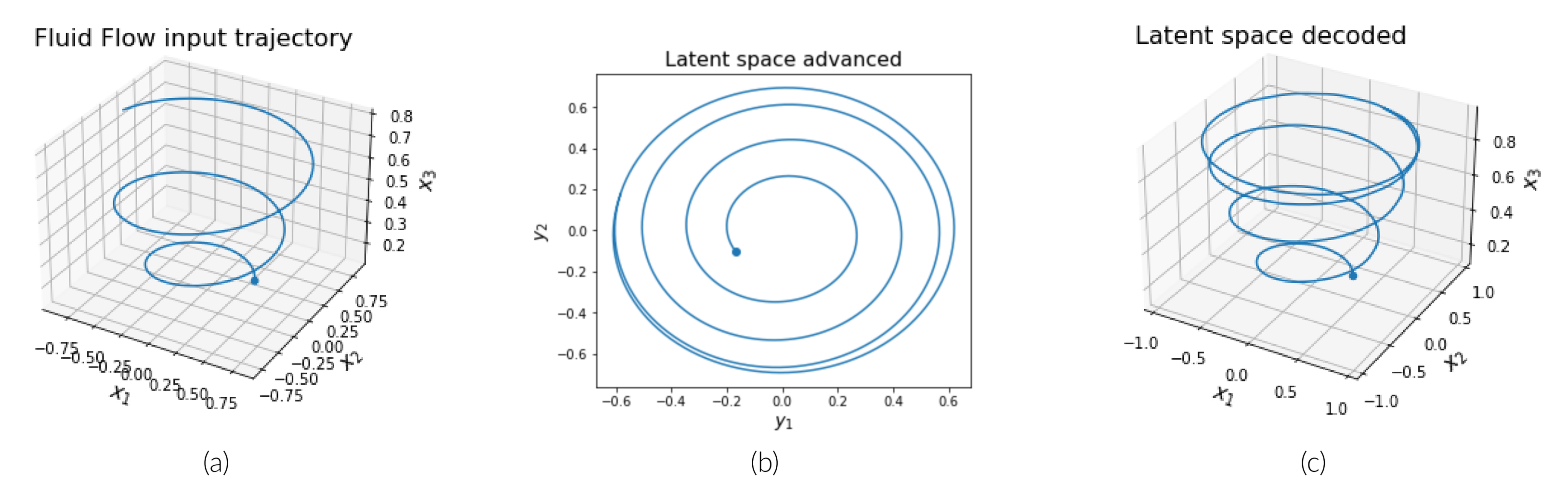

Figure 2: (a) An example validation trajectory. (b) The latent space $\mathbb{R}^3 \to \mathbb{R}^2$. Input trajectory encoded and predicted out 700 steps (Δ t = 0.05). (c) The Latent space decoded. The model accurately predicted the trajectory as it approaches its limit cycle.

Pendulum- nonlinear continuous spectra systems, described by $\dot{x_1} = x_2$ and $\dot{x_2} = -\sin(x_1)$. $x_1 = \theta$ is the angle displacement and $x_2 = \dot{\theta}$ is the angular velocity.

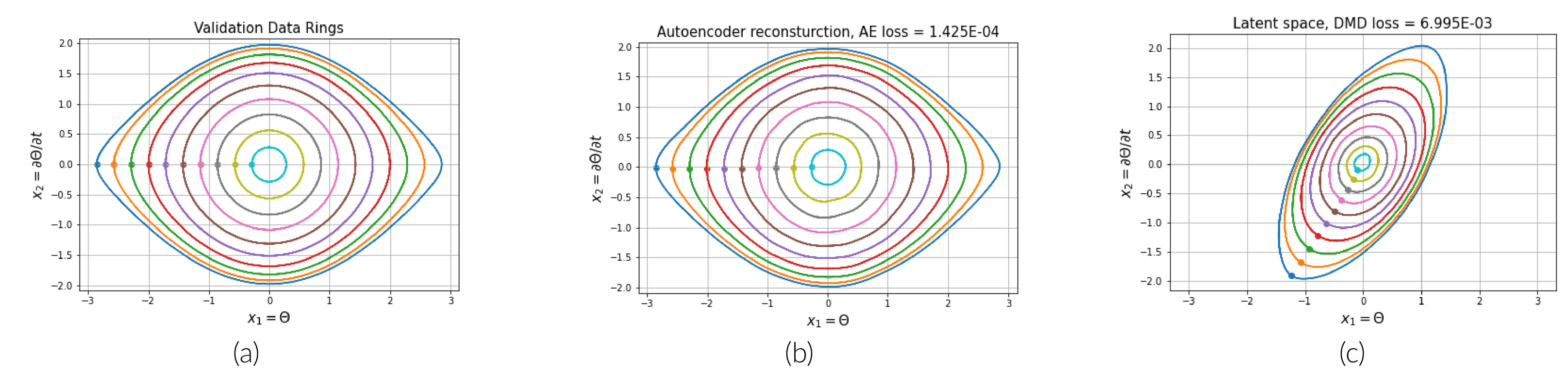

Figure 3: (a) Subset of the pendulum validation dataset. (b) The autoencoder reconstruction to ensure that g is invertible. (c) The latent space (encoded trajectories). In the latent space coordinates, the trajectories exhibit approximately linear dynamics. The model reduced the linearity loss L_2 by a factor of $1E2$.

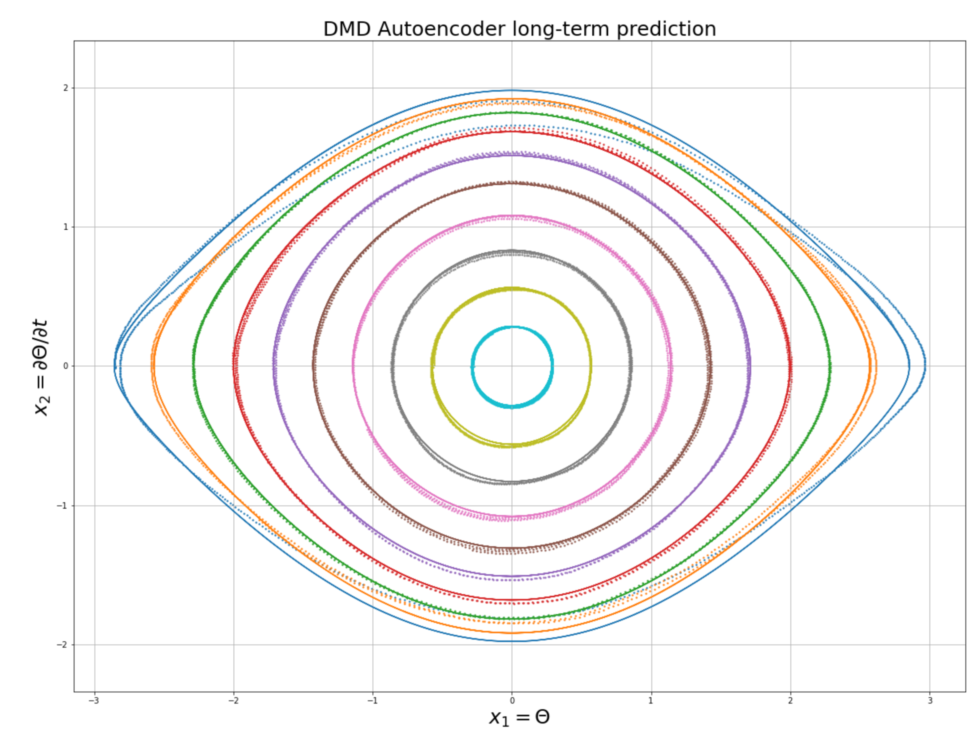

Figure 4: The simple DMD autoencoder model attempts to predict the next 1500 steps (Δ t = 0.02). When the initial angular displacement of the pendulum is small enough such that the small angle approximation holds, the model accurately predicts long-term future states. Otherwise, the predicted trajectories remain closed yet accumulated error is introduced.

Summary and Discussion

In this report, we have developed the simple DMD autoencoder model and analyzed its performance on time series generated by nonlinear dynamical systems. The model successfully learns a nonlinear mapping g where the embedded dynamics are approximately linear. The proposed model can globally enhance long-term predictions of time series. Next research steps are to test this model on a dataest coming from a strongly nonlinear, high dimensional dynamical system or from experiment.

Code Availability

The code is available at `https://github.com/opaliss/dmd_autoencoder`

References

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