

Optimization for Machine Learning

CS-439

Lecture 6: Non-convex optimization

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Trajectory Analysis

Even if the “landscape” (graph) of a nonconvex function has local minima, saddle points, and flat parts, gradient descent may avoid them and still converge to a global minimum.

For this, one needs a good starting point and some theoretical understanding of what happens when we start there—this is **trajectory analysis**.

2018: trajectory analysis for training deep **linear** linear neural networks, under suitable conditions [ACGH18].

Here: vastly simplified setting that allows us to show the main ideas (and limitations).

Linear models with several outputs

Recall: Learning linear models

- ▶ n inputs $\mathbf{x}_1, \dots, \mathbf{x}_n$, where each input $\mathbf{x}_i \in \mathbb{R}^d$
- ▶ n outputs $y_1, \dots, y_n \in \mathbb{R}$
- ▶ Hypothesis (after centering):

$$y_i \approx \mathbf{w}^\top \mathbf{x}_i,$$

for a weight vector $\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d$ to be learned.

Now more than one output value:

- ▶ n outputs $\mathbf{y}_1, \dots, \mathbf{y}_n$, where each output $\mathbf{y}_i \in \mathbb{R}^m$
- ▶ Hypothesis:

$$\mathbf{y}_i \approx W \mathbf{x}_i,$$

for a weight matrix $W \in \mathbb{R}^{m \times d}$ to be learned.

Minimizing the least squares error

Compute

$$W^* = \operatorname{argmin}_{W \in \mathbb{R}^{m \times d}} \sum_{i=1}^n \|W \mathbf{x}_i - \mathbf{y}_i\|^2.$$

- ▶ $X \in \mathbb{R}^{d \times n}$: matrix whose columns are the \mathbf{x}_i
- ▶ $Y \in \mathbb{R}^{m \times n}$: matrix whose columns are the \mathbf{y}_i

Then

$$W^* = \operatorname{argmin}_{W \in \mathbb{R}^{m \times d}} \|WX - Y\|_F^2,$$

where $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$ is the **Frobenius norm** of a matrix A .

Frobenius norm of A = Euclidean norm of $\operatorname{vec}(A)$ (“flattening” of A)

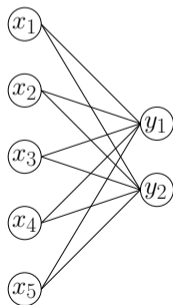
Minimizing the least squares error II

$$W^* = \operatorname{argmin}_{W \in \mathbb{R}^{m \times d}} \|WX - Y\|_F^2$$

is the global minimum of a convex quadratic function $f(W)$.

To find W^* , solve $\nabla f(W) = \mathbf{0}$ (system of linear equations).

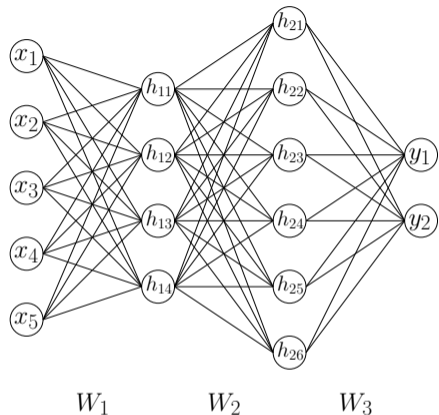
\Leftrightarrow training a **linear neural network with one layer** under least squares error.



W

$$\mathbf{x} \mapsto \mathbf{y} = W\mathbf{x}$$

Deep linear neural networks



$$\mathbf{x} \mapsto \mathbf{y} = W_3 W_2 W_1 \mathbf{x}$$

Not more expressive:

$$\mathbf{x} \mapsto \mathbf{y} = W_3 W_2 W_1 \mathbf{x} \quad \Leftrightarrow \quad \mathbf{x} \mapsto \mathbf{y} = W \mathbf{x}, \quad W := W_3 W_2 W_1.$$

Training deep linear neural networks

With ℓ layers:

$$W^* = \operatorname{argmin}_{W_1, W_2, \dots, W_\ell} \|W_\ell W_{\ell-1} \cdots W_1 X - Y\|_F^2,$$

Nonconvex function for $\ell > 1$.

Simple playground in which we can try to understand why training deep neural networks with gradient descent works.

Here: all matrices are 1×1 , $W_i = x_i$, $X = 1$, $Y = 1$, $\ell = d \Rightarrow f : \mathbb{R}^d \rightarrow \mathbb{R}$,

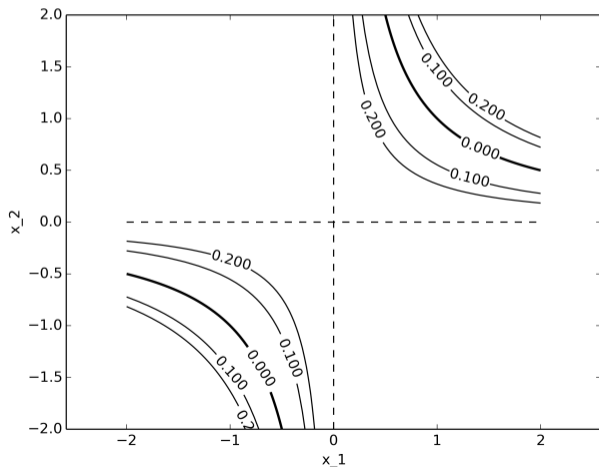
$$f(\mathbf{x}) := \frac{1}{2} \left(\prod_{k=1}^d x_k - 1 \right)^2.$$

Toy example in our simple playground.

But analysis of gradient descent on f has similar ingredients as the one on general deep linear neural networks [ACGH18].

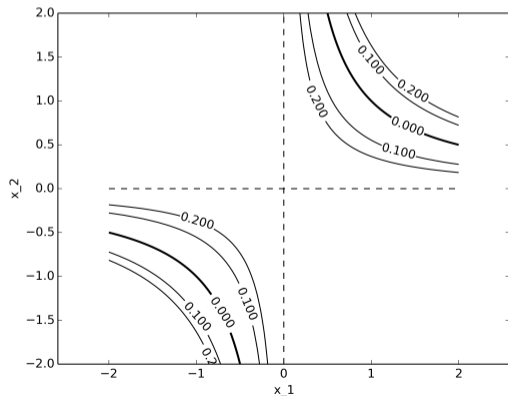
A simple nonconvex function

As d is fixed, abbreviate $\prod_{k=1}^d x_k$ by $\prod_k x_k$: $f(\mathbf{x}) = \frac{1}{2} \left(\prod_k x_k - 1 \right)^2$



The gradient

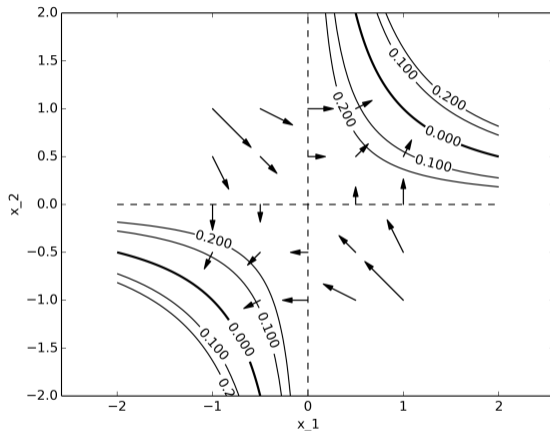
$$\nabla f(\mathbf{x}) = \left(\prod_k x_k - 1 \right) \left(\prod_{k \neq 1} x_k, \dots, \prod_{k \neq d} x_k \right).$$



Critical points ($\nabla f(\mathbf{x}) = \mathbf{0}$):

- ▶ $\prod_k x_k = 1$ (global minima)
 - ▶ $d = 2$: the hyperbola $\{(x_1, x_2) : x_1 x_2 = 1\}$
- ▶ at least **two** of the x_k are zero (saddle points)
 - ▶ $d = 2$: the origin $(x_1, x_2) = (0, 0)$

Negative gradient directions (followed by gradient descent)



Difficult to avoid convergence to a global minimum, but it is possible (Exercise 42).

Convergence analysis: Overview

Want to show that for any $d > 1$, and from [anywhere](#) in $X = \{\mathbf{x} : \mathbf{x} > \mathbf{0}, \prod_k \mathbf{x}_k \leq 1\}$, gradient descent will converge to a global minimum.

f is not smooth over X . We show that f is smooth along the trajectory of gradient descent for suitable L , so that we get sufficient decrease

$$f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t) - \frac{1}{2L} \|\nabla f(\mathbf{x}_t)\|^2, \quad t \geq 0.$$

Then, we cannot converge to a saddle point: all these have (at least two) zero entries and therefore function value $1/2$. But for starting point $\mathbf{x}_0 \in X$, we have $f(\mathbf{x}_0) < 1/2$, so we can never reach a saddle while decreasing f .

Doesn't this imply converge to a global minimum? No!

- ▶ Sublevel sets are unbounded, so we could in principle run off to infinity.
- ▶ Other bad things might happen (we haven't characterized what can go wrong).

Convergence analysis: Overview II

For $\mathbf{x} > \mathbf{0}$, $\prod_k \mathbf{x}_k \geq 1$, we also get convergence (Exercise 41).

\Rightarrow convergence from anywhere in the interior of the **positive orthant** $\{\mathbf{x} : \mathbf{x} > \mathbf{0}\}$.

But there are also starting points from which gradient descent will not converge to a global minimum (Exercise 42).

Main tool: Balanced iterates

Definition

Let $\mathbf{x} > \mathbf{0}$ (componentwise), and let $c \geq 1$ be a real number. \mathbf{x} is called *c-balanced* if $x_i \leq cx_j$ for all $1 \leq i, j \leq d$.

Any initial iterate $\mathbf{x}_0 > \mathbf{0}$ is *c-balanced* for some (possibly large) c .

Lemma

Let $\mathbf{x} > \mathbf{0}$ be *c-balanced* with $\prod_k x_k \leq 1$. Then for any stepsize $\gamma > 0$, $\mathbf{x}' := \mathbf{x} - \gamma \nabla f(\mathbf{x})$ satisfies $\mathbf{x}' \geq \mathbf{x}$ (componentwise) and is also *c-balanced*.

Proof.

$$\Delta := -\gamma(\prod_k x_k - 1)(\prod_k x_k) \geq 0. \quad \nabla f(\mathbf{x}) = (\prod_k x_k - 1) \left(\prod_{k \neq 1} x_k, \dots, \prod_{k \neq d} x_k \right).$$

Gradient descent step:

For i, j , we have $x_i \leq cx_j$ and $x_j \leq cx_i$
($\Leftrightarrow 1/x_i \leq c/x_j$). We therefore get

$$x'_k = x_k + \frac{\Delta}{x_k} \geq x_k, \quad k = 1, \dots, d.$$

$$x'_i = x_i + \frac{\Delta}{x_i} \leq cx_j + \frac{\Delta c}{x_j} = cx'_j.$$

□

Bounded Hessians along the trajectory

Compute $\nabla^2 f(\mathbf{x})$:

$\nabla^2 f(\mathbf{x})_{ij}$ is the j -th partial derivative of the i -th entry of $\nabla f(\mathbf{x})$.

$$(\nabla f)_i = \left(\prod_k x_k - 1 \right) \prod_{k \neq i} x_k$$

$$\nabla^2 f(\mathbf{x})_{ij} = \begin{cases} \left(\prod_{k \neq i} x_k \right)^2, & j = i \\ 2 \prod_{k \neq i} x_k \prod_{k \neq j} x_k - \prod_{k \neq i, j} x_k, & j \neq i \end{cases}$$

Need to bound $\prod_{k \neq i} x_k$, $\prod_{k \neq j} x_k$, $\prod_{k \neq i, j} x_k$!

Bounded Hessians along the trajectory II

Lemma

Suppose that $\mathbf{x} > \mathbf{0}$ is c -balanced. Then for any $I \subseteq \{1, \dots, d\}$, we have

$$\left(\frac{1}{c}\right)^{|I|} \left(\prod_k x_k\right)^{1-|I|/d} \leq \prod_{k \notin I} x_k \leq c^{|I|} \left(\prod_k x_k\right)^{1-|I|/d}.$$

Proof.

For any i , we have $x_i^d \geq (1/c)^d \prod_k x_k$ by balancedness, hence $x_i \geq (1/c)(\prod_k x_k)^{1/d}$. It follows that

$$\prod_{k \notin I} x_k = \frac{\prod_k x_k}{\prod_{i \in I} x_i} \leq \frac{\prod_k x_k}{(1/c)^{|I|} (\prod_k x_k)^{|I|/d}} = c^{|I|} \left(\prod_k x_k\right)^{1-|I|/d}.$$

The lower bound follows in the same way from $x_i^d \leq c^d \prod_k x_k$. □

Bounded Hessians along the trajectory III

Lemma

Let $\mathbf{x} > \mathbf{0}$ be c -balanced with $\prod_k x_k \leq 1$. Then

$$\|\nabla^2 f(\mathbf{x})\| \leq \|\nabla^2 f(\mathbf{x})\|_F \leq 3dc^2.$$

where $\|A\|_F$ is the Frobenius norm and $\|A\|$ the spectral norm.

Proof.

$\|A\| \leq \|A\|_F$: Exercise 43. Now use previous lemma and $\prod_k x_k \leq 1$:

$$|\nabla^2 f(\mathbf{x})_{ii}| = |(\prod_{k \neq i} x_k)^2| \leq c^2$$

$$|\nabla^2 f(\mathbf{x})_{ij}| \leq |2 \prod_{k \neq i} x_k \prod_{k \neq j} x_k| + | \prod_{k \neq i, j} x_k| \leq 3c^2.$$

Hence, $\|\nabla^2 f(\mathbf{x})\|_F^2 \leq 9d^2c^4$. Taking square roots, the statement follows. □

Smoothness along the trajectory

Lemma

Let $\mathbf{x} > \mathbf{0}$ be c -balanced with $\prod_k x_k < 1$, $L = 3dc^2$. Let $\gamma := 1/L$. Then for all $0 \leq \nu \leq \gamma$,

$$\mathbf{x}' := \mathbf{x} - \nu \nabla f(\mathbf{x}) \geq \mathbf{x}$$

is c -balanced with $\prod_k x'_k \leq 1$, and f is smooth with parameter L over the line segment connecting \mathbf{x} and $\mathbf{x} - \gamma \nabla f(\mathbf{x})$.

Proof.

- ▶ $\mathbf{x}' \geq \mathbf{x} > \mathbf{0}$ is c -balanced by Lemma 6.5.
- ▶ $\nabla f(\mathbf{x}) \neq \mathbf{0}$ (due to $\mathbf{x}' > \mathbf{0}$, $\prod_k x_k < 1$, we can't be at a critical point).
- ▶ No overshooting: we can't reach $\prod_k x'_k = 1$ (global minimum) for $\nu < \gamma$, as f is smooth with parameter L between \mathbf{x} and \mathbf{x}' (using previous bound on Hessians in Lemma 6.1).
- ▶ By continuity, $\prod_k x'_k \leq 1$ for all $\nu \leq \gamma$.
- ▶ f is smooth with parameter L between \mathbf{x} and \mathbf{x}' for $\nu = \gamma$.

Convergence

Theorem

Let $c \geq 1$ and $\delta > 0$ such that $\mathbf{x}_0 > \mathbf{0}$ is c -balanced with $\delta \leq \prod_k (\mathbf{x}_0)_k < 1$. Choosing stepsize

$$\gamma = \frac{1}{3dc^2},$$

gradient descent satisfies

$$f(\mathbf{x}_T) \leq \left(1 - \frac{\delta^2}{3c^4}\right)^T f(\mathbf{x}_0), \quad T \geq 0.$$

- ▶ Error converges to 0 exponentially fast.
- ▶ Exercise 44: iterates themselves converge (to an optimal solution).

Convergence: Proof

Proof.

- ▶ For $t \geq 0$, f is smooth between \mathbf{x}_t and \mathbf{x}_{t+1} with parameter $L = 3dc^2$.
- ▶ Sufficient decrease:

$$f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t) - \frac{1}{6dc^2} \|\nabla f(\mathbf{x}_t)\|^2.$$

For every c -balanced \mathbf{x} with $\delta \leq \prod_k x_k \leq 1$, $\|\nabla f(\mathbf{x})\|^2$ equals

$$2f(\mathbf{x}) \sum_{i=1}^d \left(\prod_{k \neq i} x_k \right)^2 \geq 2f(\mathbf{x}) \frac{d}{c^2} \left(\prod_k x_k \right)^{2-2/d} \geq 2f(\mathbf{x}) \frac{d}{c^2} \left(\prod_k x_k \right)^2 \geq 2f(\mathbf{x}) \frac{d}{c^2} \delta^2.$$

- ▶ Hence, $f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t) - \frac{1}{6dc^2} 2f(\mathbf{x}_t) \frac{d}{c^2} \delta^2 = f(\mathbf{x}_t) \left(1 - \frac{\delta^2}{3c^4} \right)$.

Discussion

Fast convergence as for strongly convex functions!

But there is a catch...

Consider starting point $\mathbf{x}_0 = (1/2, \dots, 1/2)$.

$$\delta \leq \prod_k (\mathbf{x}_0)_k = 2^{-d}.$$

Decrease in function value by a factor of

$$\left(1 - \frac{1}{3 \cdot 4^d}\right),$$


per step.

Need $T \approx 4^d$ to reduce the initial error by a constant factor not depending on d .

Problem: gradients are exponentially small in the beginning, extremely slow progress.

For polynomial runtime, must start at distance $O(1/\sqrt{d})$ from optimality.

Bibliography

-  Sanjeev Arora, Nadav Cohen, Noah Golowich, and Wei Hu.
A convergence analysis of gradient descent for deep linear neural networks.
CoRR, [abs/1810.02281](https://arxiv.org/abs/1810.02281), 2018.